Value at Risk (VaR)

Linear model for VaR

$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i$$

where we have a portfolio consisting of ′n′ assets and also

ΔP=rupee change in the portfolio value in one day.

α_i=amount invested in asset ′i′.

Δx_i=change in return on asset ′i′ in one day.

Value at Risk (VaR)

We can use the linear model, i.e.,

$\Delta P = \alpha_1 \Delta x_1 + \alpha_2 \Delta x_2 + \ldots\ldots + \alpha_n \Delta x_n$

to calculate the VaR of a portfolio consisting of more than one asset. From this linear model we know what will be the standard deviation, i.e., risk of the portfolio per day. Knowing that we can find the N day VaR at 99% confidence level. Similarly we can calculate the N day VaR for 95% or 90% confidence level.

Value at Risk (VaR)

Remember that the variance for the portfolio is given by the following formulae:

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\alpha_i\alpha_j\sigma_i\sigma_j$$

Value at Risk (VaR)

Duration: It means how long, on an average, the holder of an investment/bond/portfolio has to wait before receiving the cash payments. Its unit is in days/weeks/years depending what unit we are using.

Now we know that

$B = c_1 \exp\{-yt_1\} + c_2 \exp\{-yt_2\} + \ldots.. + c_n \exp\{-yt_n\}$

where:

B = NPV of the investment/bond price, etc., what ever it signifies.

c_i = payments made at the time period i.

y = continuously compounded interest rate (assumed to be constant throughout).

t_i = time period i when the payment of c_i is made.

Value at Risk (VaR)

From this we can calculate the duration, which is given by

$D = [t_1 c_1 \exp\{-yt_1\} + t_2 c_2 \exp\{-yt_2\} + \ldots + t_n c_n \exp\{-yt_n\}]/B$

Simple mathematics will show that

$\Delta B = -DB\Delta y$

This means that the percentage change in the investment/bond/portfolio value is equal to its duration multiplied by the change of the interest rate, i.e., how much the yield curve has shifted.

Value at Risk (VaR)

Considering we have a portfolio we can similarly find the change in the portfolio value and then the corresponding VaR at a given confidence level.

Hence we have

$\sigma_P = D*P*\sigma_y$

where

σ_P = volatility of the portfolio

σ_y = volatility of the interest rate per day

Value at Risk (VaR)

Consider you have a bond portfolio whose worth is Rs. 10 Lakhs and the duration is 2 years. Also assume that the daily volatility is 0.10%.

Hence we have:

$\sigma_P = 2*10*10^5*0.001 = 2000$

- Hence VaR (at 99% confidence level) for a month, i.e., 30 days, is $2.33*2000*30^{1/2} = 25524$
- Hence VaR (at 95% confidence level) for a month, i.e., 30 days, is $1.96*2000*30^{1/2} = 23662$
- Hence VaR (at 90% confidence level) for a month, i.e., 30 days, is $1.65*2000*30^{1/2} = 18075$

Value at Risk (VaR)

An alternative definition for volatility is the standard deviation of $\Delta y/y$. In that case we have

$\sigma_P = D*P*y*\sigma_y$

Similar calculation will give us different values of VaR considering different confidence levels.

Value at Risk (VaR)

The use of linear model is most effective in giving the true value of the VaR for a portfolio which consists of no derivatives. The assets in the portfolio can be stocks, foreign exchange, commodities, bonds, etc. In that case, the change in the portfolio value is dependent on stock prices, exchange rates, commodity prices, zero coupon rate, etc.

The linear model is only an approximation when we have options in the portfolio.

Thus we know that $\delta = \Delta P / \Delta S$

where

δ = delta of the portfolio for the option position of a stock.

S = current price of the same stock on which the option is based and this option is a part of the portfolio we are considering.

Also remember $\Delta x = \Delta S / S$

Thus:

$\Delta P = S_1 \delta_1 \Delta x_1 + S_2 \delta_2 \Delta x_2 + \ldots.. + S_n \delta_n \Delta x_n$

Value at Risk (VaR)

Consider we have a portfolio consisting of options on HLL and ITC. The respective delta values are 5000 and 7500. The present stock prices are Rs.350 and Rs.300 respectively. Also assume the daily volatility of prices are 5% and 7% respectively and the correlation coefficient between the prices of the two stocks is 0.75.

Then:

$\sigma_P=\{(350*0.05)^2+(300*0.07)^2+2*0.75*350*300*0.05*0.07\}^{1/2}=36$

For calculating the VaR value (at 95% confidence level) for 10 days, we have:

$VaR=1.65*10^{1/2}*36=188$

Similar calculations will yield VaR (at 99% confidence level), VaR (at 90% confidence level).

Value at Risk (VaR)

Remember that even though the asset price is normally distributed (an assumption, but can be taken as valid as the results obtained are well within limits of the errors), we have positive (negative) skewed value of returns for the long (short) call on the same asset.

Value at Risk (VaR)

A more accurate value of VaR can be found using the respective delta and gamma measures of the assets. Remember these values give a relationship between the change in portfolio value and the changes in the return of the respective assets, which form the portfolio.

Thus for a single asset with a price of S we have the change in the value of the portfolio given as $\Delta P = S*\delta*\Delta x + 0.5*S^2*\gamma*\Delta x^2$

Remember here the distribution of ΔP is non-normal, distribution of Δx is normal (with mean zero and standard deviation of σ). To derive the above formulae we use the Taylors's series expansion.

Value at Risk (VaR)

For a portfolio with ′n′ number of assets and with each instrument in the portfolio being **dependent on only one of the market variables**, then the general formulae to calculate the change in the portfolio value is given by:

$$\Delta P = S_1 * \delta_1 * \Delta x_1 + S_2 * \delta_2 * \Delta x_2 + \ldots + S_n * \delta_n * \Delta x_n + 0.5 * S_1^2 * \gamma_1 * \Delta x_1^2 + 0.5 * S_2^2 * \gamma_2 * \Delta x_2^2 + \ldots + 0.5 * S_n^2 * \gamma_n * \Delta x_n^2$$

Where:

$S_i = i^{th}$ market variable

δ_i = delta of the portfolio wrt the i^{th} market variable

γ_i = gamma of the portfolio wrt the i^{th} market variable

Value at Risk (VaR)

For a portfolio with ′n′ number of assets the general formulae for finding out the change in the portfolio value is

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j$$

where γ_{ij} is the **cross gamma** given by the formulae

$$\gamma_{ij} = \frac{\partial^2 P}{\partial S_i \partial S_j} = \frac{\partial^2 P}{\partial S_j \partial S_i}$$

Value at Risk (VaR)

Calculating the change in the portfolio value and hence the volatility of the same is difficult if we have ′n′ number of assets and their corresponding options, as the exact mathematical formulation for the following equation is difficult.

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j$$

Value at Risk (VaR)

Credit for inventing the Monte Carlo method often goes to Stanislaw Ulam, a Polish born mathematician who worked for John von Neumann on the United States′ Manhattan Project during World War II. Ulam is primarily known for designing the hydrogen bomb with Edward Teller in 1951. He invented the Monte Carlo method in 1946 while pondering the probabilities of winning a card game of solitaire.

The Monte Carlo method, as it is understood today, encompasses any technique of statistical sampling employed to approximate solutions to quantitative problems.

Value at Risk (VaR)

To calculate VaR using Monte Carlo simulation we follow the steps described below:

1) Find the value of the portfolio today, i.e., $P_{t=0}$.
2) Sample once from the multivariate distributions of the Δx_i.
3) Use the Δx_i to determine market variables at the end of today.
4) Revaluate the portfolio at the end of day, i.e., $P_{t=1}$.

Value at Risk (VaR)

5) Calculate ΔP, using the differences between the portfolio value at the beginning of the day and at the end of the day, i.e., $P_{t=1}$-$P_{t=0}$.
6) Repeat steps # 1-5 many times to build up a probability distribution for ΔP.
7) VaR is the appropriate fractile of the distribution times square root of N (number of days).

Value at Risk (VaR)

One major **disadvantage** of the Monte Carlo Simulation methodology is that we **assume the market variables to be normally distributed**.

To over come this problem we use **historical simulation**. In historical simulation we first create a data base consisting of the daily movement of all market variables over the years.

Value at Risk (VaR)

1) In the 1st simulation trial the percentage changes in each market variable is assumed to be equal to that of the market variables as in the first day covered by the database.
2) In the 2nd simulation trial the percentage changes in each market variable is assumed to be equal to that of the market variables as in the second day covered by the database.

…….

n) In the nth simulation trial the percentage changes in each market variable is assumed to be equal to that of the market variables as in the nth day covered by the database.

n+1) Using the information from each step we find the ΔP and hence the distribution of ΔP.

n+2) From the distribution of ΔP we calculate the appropriate percentile of the probability distribution for ΔP and finally the appropriate VaR

Value at Risk (VaR)

The main disadvantages of historical simulation methodology are:

- No stress testing is possible.
- No sensitivity analysis is possible.
- No back testing is possible.

Value at Risk (VaR)

Stress Testing

This involves testing how well a portfolio performs under some of the most extreme market moves seen in the last 10 to 20 years.

Back Testing

1) Tests how well VaR estimates would have performed in the past.
2) We could ask the question, about how often was the actual 10-day loss greater than the 99% 10 day VaR?

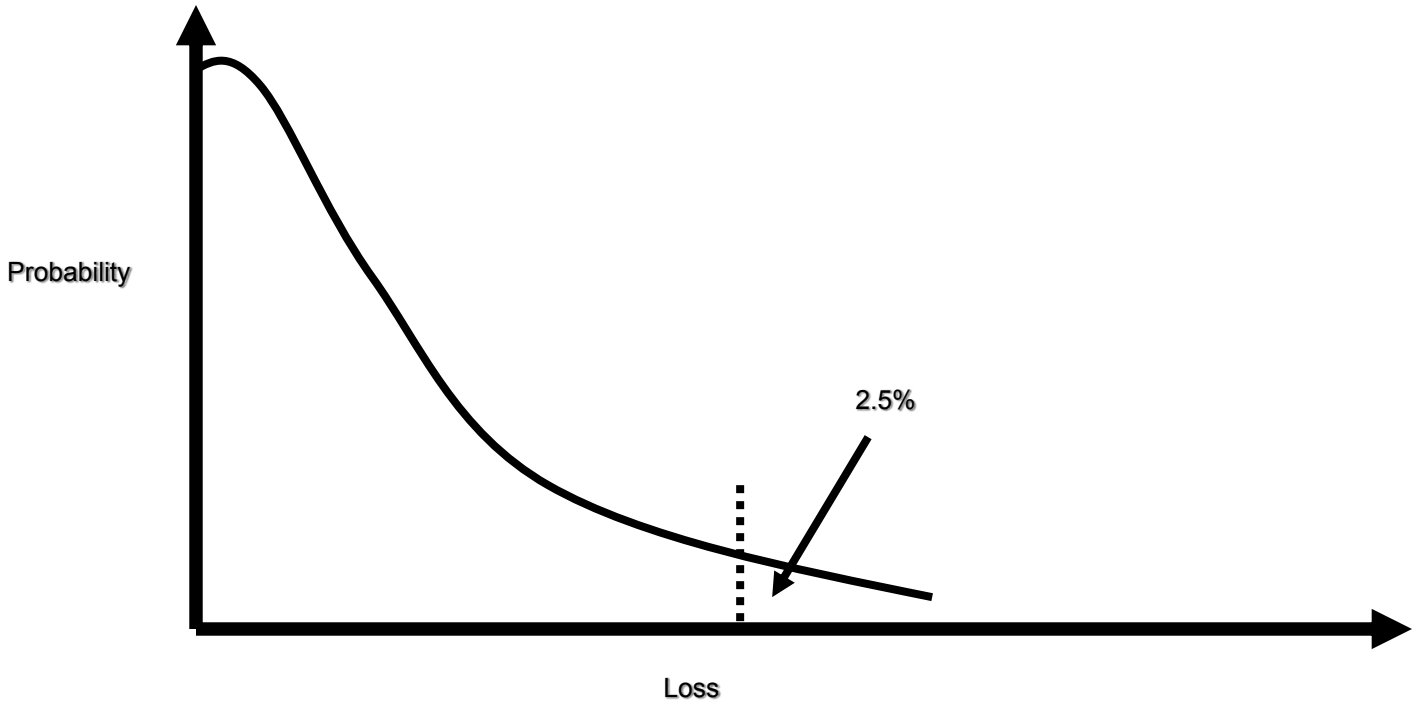
Value at Risk (VaR)

We usually use Principal Components Analysis to handle risk arising from groups of highly correlated market variables. Using the Principal Component Analysis we try to define a set of factors that can be taken as the market variables.

Value at Risk (VaR)

Remember a well known characteristic of the loss distributions is that they have fat tails. Fat tails are the extreme sections of the distributions, and indicate that large losses, although unlikely because their probabilities remain low, still they have some likelihood to occur that is not negligible. The fatness of the tail refers to non-zero probabilities over the long end of the distributions. Now remember that for a normal distribution, VaR for a 2.5% one tailed confidence level corresponds to a multiple of loss volatility of 1.96. Therefore if the loss volatility is 100, the unexpected loss will not exceed the upper bound of 196 in more than two or three cases out of 100 scenarios. But if the distribution is not normal then such simple calculations do not apply.

Value at Risk (VaR)



Value at Risk (VaR)

$P[X \leq x] = p[\{X-\mu\}/\sigma \leq \{x-\mu\}/\sigma] = 1-0.025 = 0.975$

Thus $P[Z \leq z] = 0.975$, hence $z = 1.96$

As $\mu = 0$, hence $x = 1.96*\sigma$. But remember that for distributions which are not normal, this calculation will not hold.

In case you have the distribution as exponential, then also we can find the cumulative probability of default, or the risk exposure depending on the value of the loss one can sustain. Hence using a value of α, we can find x_1, which is the loss one can sustain.

$$\int_0^x \exp(-x)dx = \alpha$$

Value at Risk (VaR)

When implementing techniques, based on confidence level (CL) and loss percentiles, there is a need for common benchmarks, such as CL, for all the players must be the same. With a very high CL, VaR could be so high that business transactions would soon become limited. On the other hand if the benchmark is different then the banks have different standards for evaluating their VaR. In other words a level playing field should be agreed upon, so that there is no confusion in this respect.

Note, the normal distribution is an approximation for losses in general, over a short period of time, but it cannot be applied for credit risk, for which the loss distribution is highly asymmetric.

In spite of repeating the same yet it is important to note that for any FI it is important to find the distribution of the (i) inflows and (ii) outflows and then try to find the resulting distribution for the net income/loss so that we can find the VaR at a given confidence level and for the particular number of days, as stipulated.

Value at Risk (VaR)

$P(A \text{ and } B) = \rho_{AB} * \{P(A) * [1 - P(A)]\}^{1/2} * \{P(B) * [1 - P(B)]^{1/2} + P(A) * P(B)$

Where:

A = default of the underlying obligor.

B = default by the guarantor or protection seller.

P(A and B) = probability that both A and B default.

P(A) = probability of default of underlying obligor (assuming **binomial distribution**).

P(B) = probability of default of guarantor (assuming **binomial distribution**).

ρ_{AB} = correlation between the default events A and B.

Value at Risk (VaR)

In general, the variables that have impact on the intrinsic value of a credit derivative are the same variables used to value a credit risk asset. The following are some of the variables:

1) Likely probability of default of the underlying credit and the protection seller.
2) Pessimistic probability of default.
3) Joint probability of default of the underlying credit and the protection of the seller.
4) Timing of default.
5) Recovery rate distribution.
6) Interest rate.
7) Timing of recovery.

Value at Risk (VaR)

Default rates are unstable as time passes. The volatility of yearly historical default rate is the standard deviation of observed time series and volatility increases with default rate. Volatility also provides a simple way to measure unexpected loss for a loan portfolio and this expected loss is **proportional** to the average default rate.